Supersymmetric deformations of F1-NS5-branes and their exact CFT description

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Abstract: We consider certain classes of operators in the exact conformal field theory $SL(2,\mathbb{R}) \times SU(2) \times U(1)^4$ describing strings in an $AdS_3 \times S^3 \times \mathbb{T}^4$ geometry supported by Neveu-Schwarz 3-form fluxes. This background arises in the near-horizon limit of a system of NS5-branes wrapped on a 4-torus and F1-branes smeared on the 4-torus when both types of branes are located at the same point in their common transverse space. We find a class of operators that lead to spacetime supersymmetric deformations. It is remarkable that most of these operators are not chiral primary with respect to the $\mathcal{N} = 2$ superconformal algebra on the worldsheet. A subset of these worldsheet conformal field theory deformations admits an interpretation either as a geometric deformation of the brane system or as a deformation of the distribution of the F1-branes, viewed as smooth instantons, inside the wrapped NS5-brane worldvolume. The 2-dimensional conformal field theory, however, seems to lack operators corresponding to arbitrary NS5-brane deformations, in contrast to pure NS5-brane systems where all geometric deformations can be accounted for by chiral primary operators.

Contents

1 Introduction

Objects charged under the NSNS antisymmetric tensor field of string theory, i.e. the electrically charged F1-branes and the magnetically charged NS5-branes as well as their bound states, are of particular importance since the corresponding string theory backgrounds may in principle admit an exact conformal field theory (CFT) description. In such cases the physics of these objects is amenable to the powerful methods of CFT.

The most well-known example is that of a configuration of parallel and coincident NS5-branes. Their backreaction leads to a characteristic throat-like geometry whose near-horizon limit comprises of a linear dilaton along with a 3-sphere, both of which can be described in terms of exact CFTs [1]. Another example is provided by a circular distribution of NS5-branes. This system, after an appropriate T-duality, admits a CFT description in

terms of the cosets $\mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ [2] and can be thought of as a deformation of the first configuration that resolves the strong coupling singularity associated with the linear dilaton [3, 4].

An interesting feature of these systems is that the little string theories (LSTs) that reside on the worldvolume of the NS5-branes [5] can be described holographically in terms of the associated CFTs [6]. A fundamental aspect of these holographic dualities is the existence of a dictionary between deformations of the branes described via perturbations of the original supergravity solution, parametrized by vacuum expectation values of scalar fields on the branes, and exactly marginal deformations of the underlying CFT. Such a dictionary was discussed in detail in [4] and was tested successfully in [7], by matching directly the supergravity deformations realized in the σ model description of the theory to CFT operators.

The latter analysis was motivated by earlier work [8] where it was explicitly shown that the continuous deformation of the circular NS5-brane distribution into an elliptic one was driven by a marginal perturbation of the $\mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ worldsheet σ model. The deformation of the circle into an ellipsis is one particular mode among an infinite consisting of battered circles with $n \in \mathbb{N}$ bumps distributed with $\mathbb{Z}_n$ symmetry around the original circle. These types of deformations as well as their corresponding CFT operators based on parafermions provided actually the testing ground for [7].

One interesting aspect of such deformations is related to their supersymmetry properties. Since they are realized in terms of changes of the transverse distribution of the branes, they should preserve an amount of supersymmetry and, therefore, this property should also be manifest in the CFT operators. An analysis in this spirit was performed in [9] for the case of the pointlike system of branes, whose CFT description involves the linear dilaton theory $\mathbb{R}_\phi$ and the $\mathrm{SU}(2)$ Wess-Zumino-Witten (WZW) model.

The purpose of the present paper is to analyze aspects of the interplay between space-time deformations and the corresponding marginal CFT operators, in particular with respect to their supersymmetry properties, in a third example of a system with a known exact CFT description. This system is comprised of a set of NS5- and F1-branes located at the same point in their common transverse space with four of the Euclidean worldvolume directions of the NS5-branes wrapped on a 4-torus, along which the F1-branes are smeared homogeneously. In the near-horizon limit it features a constant dilaton and a geometry of the form $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$ supported by appropriate NSNS 3-form fluxes. The exact CFT description is provided by the product of WZW models $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ along with four free compact bosons $\mathrm{U}(1)^4$ corresponding to $\mathbb{T}^4$. Notice that the analogue of LST in this case is a 2-dimensonal CFT residing on the boundary of AdS_3 which is known to be a deformation of a symmetric orbifold theory. This theory arises as the infrared limit of the super-Yang-Mills theory that lives on the common $(1+1)$-dimensional non-compact worldvolume of the branes. *In order to avoid any confusion, we emphasize that in this paper CFT will always mean the worldsheet theory underlying the F1-NS5-brane system and not the CFT on the boundary of the AdS$_3$.*

We will start by uncovering the CFT operators dual to some simple deformations of the brane system. Subsequently, we will perform a full-fledged analysis of the supersymmetry

properties of a large class of marginal operators in the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{U}(1)^4$ theory. Some of the operators we study have been analyzed in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality starting from [10]. The most interesting aspect of this analysis stems from the fact that for backgrounds of this type, i.e. which feature timelike curved geometries, and as opposed to the case of Minkowski spacetime [11], the existence of spacetime supersymmetry is not tight to $\mathcal{N} = 2$ superconformal (SCFT) invariance on the worldsheet [12]. Therefore, the set of chiral (or antichiral) primaries, which preserve automatically the $\mathcal{N} = 2$ SCFT symmetry, provides only a small subset of the operators that can lead to spacetime supersymmetric deformations.

This observation should be compared to what happens for the first two systems mentioned here. Those comprise only of NS5-branes and consequently time is a non-intracting factor in the sigma model. The $\mathcal{N} = 2$ superconformal algebra is realized in a conventional (hermitian) manner and the set of chiral and antichiral primaries captures precisely all possible geometric brane deformations [7, 9]. This no longer holds in the NS5/F1 system under investigation, where we will uncover, among others, a new class of operators whose effect on the branes is to perturb the originally homogeneous distribution of the F1-branes inside the NS5-branes. In other words, if we view the F1-branes as smeared instantons in the NS5-brane theory, turning on these operators corresponds to infinitesimal motions in the instanton moduli space.

The layout of this paper is as follows. We start in section 2 with a supergravity analysis of general F1-NS5-brane systems and discuss the exact CFT description of the pointlike setup as well as certain deformations thereof. In this section we also present the CFT operators that correspond to the deformations we have performed. In section 3 we review the construction of the spacetime supercharges of the undeformed $\mathrm{AdS}_3 \times S^3$ and subsequently we uncover the set of chiral and antichiral primaries of the worldsheet CFT as well as a large class of spacetime supersymmetry preserving operators. We discuss several issues pertaining to the potential interpretation of those operators in terms of brane deformations. Finally, in the last section we extend our analysis to a more general class of operators and we provide the brane description of a class of them that lead to supersymmetric deformations. In the appendices we have summarized our conventions on the $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ WZW models and we have provided the explicit realization of the $\mathcal{N} = 2$ superconformal algebra employed in the analysis of the chiral primaries.

2 F1-NS5-brane configurations

In this section we study the F1-NS5-brane system from the supergravity and exact conformal field theory description view points. In particular, we perform certain symmetric perturbations around the point where the exact CFT description is known and describe them in terms of WZW primaries and currents of the associated CFTs.

2.1 Generic 1/4 supersymmetric configurations

Our starting point is a 10-dimensional background metric of the form

$$ds^2 = H_1^{-1}(-dt^2 + dz^2) + H_5 dx^i dx^i + dy^a dy^a \,, \qquad i, a = 1, 2, 3, 4\,, \quad H_{1,5} = H_{1,5}(x)\,, \quad (2.1)$$

which for appropriate choices of the functions $H_{1,5}(x)$ represents the gravitational backreaction of a large collection of F1- and NS5-branes. The worldvolume of the F1-branes is spanned by $z^\mu = (t, z)$, $\mu = 0, 1$, while that of the NS5-branes by $z^\mu = (t, z)$ and y^a. We will assume that the 4-dimensional part of the NS5-brane worldvolume parametrized by y^a is wrapped on a flat 4-torus $\mathbb{T}^4$. Therefore both types of branes share a $(1+1)$-dimensional non-compact worldvolume parametrized by z^μ.

The coordinates x^i parametrize the common transverse space and are non-compact. Notice that since we assume that H_1 depends only on x^i but not on the additional transverse coordinates of the F1-branes y^a, the latter are effectively smeared homogeneously on the 4-torus. The geometry is supplemented by a dilaton field $\Phi = \Phi(x)$ as well as an NSNS 3-form field strength whose non-vanishing components are H_{ijk} and H_{tzi}. These are sourced, respectively, by the NS5- and F1-branes.

We can choose an orthonormal frame

$$e^\mu = H_1^{-1/2} dz^\mu \,, \qquad e^i = H_5^{1/2} dx^i \,, \tag{2.2}$$

from which we compute the spin connection with non-vanishing elements

$$\omega^{ij} = -\frac{1}{2} H_5^{-1} \partial^{[i} H_5 dx^{j]} \,, \qquad \omega^{\mu i} = -\frac{1}{2} H_1^{-3/2} H_5^{-1/2} \partial^i H_1 dz^\mu \,. \tag{2.3}$$

The Killing spinor equations arising by setting to zero the gravitino and dilatino supersymmetry variations are

$$\partial_\mu \epsilon + \frac{1}{4} \left(\omega_\mu^{ab} - \frac{1}{2} H_\mu{}^{ab} \right) \Gamma_{ab} \epsilon = 0 \,,$$
$$\Gamma^\mu \partial_\mu \epsilon - \frac{1}{12} H_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \epsilon = 0 \,. \tag{2.4}$$

In addition, we have to satisfy the equations of motion

$$R_{\mu\nu} - \frac{1}{4} \left(H^2 \right)_{\mu\nu} + 2 D_\mu D_\nu \Phi = 0 \,,$$
$$D_\mu \left(e^{-2\Phi} H^\mu{}_{\nu\rho} \right) = 0 \,. \tag{2.5}$$

From the dilatino equation we find the projections

$$\Gamma^{tz} \epsilon = \epsilon \,, \qquad \Gamma^{1234} \epsilon = -\epsilon \,, \tag{2.6}$$

where the first refers to the common worldvolume directions of the F1- and NS5-branes, while the second to the common transverse directions. These conditions reduce the amount of preserved supersymmetry to 1/4 of the original one. Therefore, for type II superstring theories we obtain, in the generic case, backgrounds which preserve 8 supersymmetries.

From the gravitino equation we deduce the form of the antisymmetric tensor field strength (all indices below are curved)

$$H_{tzi} = \partial_i H_1^{-1} \,, \qquad H_{ijk} = \epsilon_{ijk}{}^l \partial_l H_5 \,, \tag{2.7}$$

where the index is raised with the flat metric in $\mathbb{R}^4$. The form of the Killing spinor is $\epsilon = H_1^{-1/4}\epsilon_0$, with ϵ_0 being a constant spinor subject to the same projections as (2.6). These results, in combination with the dilatino equation, restrict the form of the dilaton to

$$e^{-2\Phi} = \frac{H_1}{H_5} \ . \tag{2.8}$$

Finally, the Bianchi identity $dH = 0$ requires that H_5 is a harmonic function, while it imposes no condition on H_1. The latter, however, must also be a harmonic function in order that the equations of motion are satisfied. Therefore we get

$$\partial_i \partial^i H_{1,5} = 0 \ . \tag{2.9}$$

The general solution of those equations is obtained from the (unit-normalized) densities $\rho_{1,5}(x)$ of F1- and NS5-branes as

$$H_{1,5}(x) = c_{1,5} \int_{\mathbb{R}^4} d\mathbf{x}' \frac{\rho_{1,5}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} \, , \tag{2.10}$$

where $c_1 = g_s^2 \alpha'^3 N_1/V_4$ and $c_5 = \alpha' N_5$. The numbers $N_{1,5}$ correspond to the total electric and magnetic NSNS charge. We focus on the near-horizon region of the branes and thereby we have dropped the constant term that in principle we could have added to the harmonic functions.

2.2 NS5- and F1-branes at a point, supersymmetry enhancement and exact CFT

The simplest configuration we can consider is that where both types of branes reside on the same point $x^i = 0$ in their common transverse space. Then

$$H_5 = \frac{c_5}{r^2} \, , \quad H_1 = \frac{c_1}{r^2} \, , \tag{2.11}$$

where $r^2 = x^i x^i$.

This configuration is particularly interesting for two reasons. First, the preserved supersymmetry is enhanced to 16 supercharges. This is basically due to the conformal flatness of the 6-dimensional non-trivial part of the 10-dimensional background and analogous to the supersymmetry enhancement that occurs when we probe the near horizon region of a D3-brane, where the original 16 supersymmetries are enhanced to 32.

Second, it is easy to see that the metric (2.1) and the antisymmetric-tensor field strength (2.7), after a change of coordinates $r = e^\phi$ and appropriate rescaling of t and z, take the form

$$\begin{aligned} ds^2 &= \alpha' N_5 \Big(e^{2\phi}(-dt^2 + dz^2) + d\phi^2 + d\Omega_3^2 \Big) + dy^i dy^i \, , \\ H &= 2\alpha' N_5 \left(\text{Vol}_{\text{AdS}_3} + \text{Vol}_{S^3} \right) \, , \end{aligned} \tag{2.12}$$

which describes the geometry of $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$, supported by appropriate NSNS fluxes. Along with the 3-form field strengths, this background admits an exact CFT description

in terms of the WZW models $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ and 4 free compact bosons $\mathrm{U}(1)^4$ corresponding to $\mathbb{T}^4$. As is evident from (2.12) the level of both cosets is set by N_5, while the number of F1-branes N_1 appears only in the value of the 6-dimensional string coupling (the constant dilaton)

$$g_{\mathrm{s}}^2 = \frac{N_5}{N_1} \, . \tag{2.13}$$

2.3 NS5-branes on a circle and F1-branes at a point

2.3.1 Identification of the marginal operators

For a system of NS5-branes it is known that besides the pointlike configuration, which admits an exact CFT description in terms of a linear dilaton $\mathbb{R}_\phi$ theory and the $\mathrm{SU}(2)$ WZW model [1], another system that admits an exact CFT description is that of a circular distribution. The corresponding CFT, after an appropriate T-duality, is an orbifold of the product of the coset models $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1) \times \mathrm{SU}(2)/\mathrm{U}(1)$ for the transverse space, times free bosons for the directions longitudinal to the NS5-branes [2].

An interesting way of thinking about the circular distribution is as a small deformation of the original pointlike setup. In CFT terms we can think of the deformed model as arising from an exactly marginal deformation of the original $\mathbb{R}_\phi \times \mathrm{SU}(2)$ theory [3, 4]. We would like to maintain this point of view and study the system of NS5-branes on a circle, this time in the presence of the F1-branes, as a deformation of the original $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{U}(1)^4$ theory that describes the setup where all branes reside at a single point.

Therefore, let us take the centers of the NS5-branes distributed on an N_5-polygon situated in the plane spanned by x_3 and x_4 inside the space transverse to the branes. We have

$$\vec{x}_p = r_0(0, 0, \cos\phi_p, \sin\phi_p), \qquad \phi_p = 2\pi \frac{p}{N_5}, \quad p = 0, 1, \ldots, N_5 - 1 \, . \tag{2.14}$$

This distribution of branes preserves an $\mathrm{SO}(2) \times \mathbb{Z}_{N_5}$ subgroup of the original $\mathrm{SO}(4)$ symmetry that is exhibited by the point-like setup. In the continuum limit the branes are distributed on a ring of radius r_0 situated in the (34)-plane and the symmetry subgroup becomes continuous, i.e. $\mathrm{SO}(2) \times \mathrm{SO}(2)$. After changing variables as [2]

$$
\begin{aligned}
x_1 &= r_0 \sinh\rho \cos\theta \cos\tau \,, & x_2 &= r_0 \sinh\rho \cos\theta \sin\tau \,, \\
x_3 &= r_0 \cosh\rho \sin\theta \cos\psi \,, & x_4 &= r_0 \cosh\rho \sin\theta \sin\psi \,,
\end{aligned}
\tag{2.15}
$$

with ranges

$$0 \leqslant \rho < \infty \,, \qquad 0 \leqslant \theta < \frac{\pi}{2} \,, \qquad 0 \leqslant \psi, \tau < 2\pi \,, \tag{2.16}$$

we find that the flat metric on $\mathbb{R}^4$ takes the form

$$dx^i dx^i = r_0^2 \left[(\sinh^2\rho + \cos^2\theta)(d\rho^2 + d\theta^2) + \sinh^2\rho \cos^2\theta \, d\tau^2 + \cosh^2\rho \sin^2\theta \, d\psi^2 \right] \,, \tag{2.17}$$

while the harmonic function describing the circular distribution of NS5-branes reads

$$H_5 = \frac{c_5}{\sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2 + r_0^2)^2 - 4r_0^2(x_3^2 + x_4^2)}} = \frac{c_5/r_0^2}{\sinh^2\rho + \cos^2\theta} \, . \tag{2.18}$$

Instead, since the F1-branes are all located at the origin, we have

$$H_1 = \frac{c_1}{r^2} = \frac{c_1/r_0^2}{\sinh^2 \rho + \sin^2 \theta} \ . \tag{2.19}$$

Then, the 6-dimensonal part of the background is[1]

$$ds_6^2 = (\sinh^2 \rho + \sin^2 \theta)(-dt^2 + dz^2) + d\rho^2 + d\theta^2 + \frac{\tan^2 \theta d\psi^2 + \tanh^2 \rho d\tau^2}{1 + \tan^2 \theta \tanh^2 \rho} \ ,$$

$$B_{tz} = \sinh^2 \rho + \sin^2 \theta \ , \qquad B_{\tau\psi} = \frac{1}{1 + \tan^2 \theta \tanh^2 \rho} \ , \tag{2.20}$$

$$e^{-2\Phi} = \frac{N_1}{N_5} \frac{\sinh^2 \rho + \cos^2 \theta}{\sinh^2 \rho + \sin^2 \theta} \ .$$

Asymptotically, for $\rho \to \infty$, this background approaches $\mathrm{AdS}_3 \times S^3$ which corresponds to the pointlike configuration discussed in subsection 2.2. The leading-order corrected metric, due to the circular distribution of the NS5-branes, is

$$ds_6^2 = d\rho^2 + e^{2\rho} dx^+ dx^- + d\Omega_3^2 + 4e^{-2\rho}(\sin^4 \theta d\psi^2 - \cos^4 \theta d\tau^2) - 2 \cos 2\theta dx^+ dx^- + \cdots , \tag{2.21}$$

where we introduced null coordinates $x^\pm = \frac{z \pm t}{2}$. The corresponding expression for the antisymmetric tensor is

$$B_{\tau\psi} = \cos^2 \theta + 4e^{-2\rho} \cos^2 \theta \sin^2 \theta + \cdots , \qquad B_{x^+ x^-} = \frac{1}{2} e^{2\rho} - \cos \theta + \cdots . \tag{2.22}$$

The first term in the deformation of the metric as well as the deformation of $B_{\tau\psi}$ originate from

$$e^{-2\rho} J^3 \bar{J}^3 \sim \Phi^{sl}_{0;-1,-1} J^3 \bar{J}^3 \ , \tag{2.23}$$

where $\Phi^{sl}_{0;-1,-1}$ is the normalizable branch of the identity operator in $\mathrm{SL}(2,\mathbb{R})$ with conformal dimension 0 and $J_3, \bar{J}_3$ are the Cartan currents of $\mathrm{SU}(2)$.[2] This is in direct analogy with the deformation of the linear dilaton $\mathbb{R}_\phi$ theory times the $\mathrm{SU}(2)$ WZW model that perturbs a system of pointlike NS5-branes towards a small circle [9].[3] Notice that in both systems (F1-NS5 and pure NS5), we could use the genuine identity operator instead of the normalizable dimension-zero one, but the corresponding marginal deformation driven by $J_3 \bar{J}_3$ would not be related to any brane displacement.

The second term in the deformation of the metric as well as the deformation of $B_{x^+ x^-}$ resides in the $\mathrm{SL}(2,\mathbb{R})$ sector of the original CFT. We can find the corresponding CFT operator by using the relations in appendices A and B. It reads

$$- \cos 2\theta e^{-4\rho} K^+ \bar{K}^+ \sim -\Phi^{su}_{1;0,0} \Phi^{sl}_{1;-2,-2} K^+ \bar{K}^+ \ . \tag{2.24}$$

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[1]As in the pointlike case we can get rid of factors of g_s, N_1, V_4 and r_0 by rescaling t and z. We will also omit the universal factor $\alpha' N_5$ to avoid cluttering of the formulas and stick to these conventions for the rest of the paper.

[2]The explicit semiclassical expressions for all WZW currents and operators that we use, can be found in appendices A and B.

[3] In that case we have a perturbation of the form $e^{-q\phi} J_3 \bar{J}_3$. By taking into account the background charge $-q/2$ of a canonically normalized boson, the conformal dimension of $e^{-q\phi}$ is zero.

Notice that due to the fact that the quantum numbers of $\Phi^{sl}_{1;-2,-2}$ correspond to the highest weight state of the negative discrete series, its OPEs with K^+ and $\bar{K}^+$ are regular and there is no normal-ordering ambiguity for the $SL(2,\mathbb{R})$ operators.

To summarize, the deformation of $SL(2,\mathbb{R}) \times SU(2)$ that corresponds to a circular configuration of NS5-branes with the F1-branes still sitting at a point, takes the form

$$\Phi^{sl}_{0;-1,-1}J^3\bar{J}^3 - \Phi^{su}_{1;0,0}\Phi^{sl}_{1;-2,-2} K^+\bar{K}^+ \ . \tag{2.25}$$

We can easily see that both of these operators are marginal.

2.4 NS5-branes at a point and F1-branes on a circle

A configuration complementary to the one studied in the previous subsection is that of NS5-branes residing at a point with the F1-branes put on a circle. In this case the appropriate coordinate system, i.e. that in which the deformation is manifestly marginal, is actually

$$x_1 = r\cos\theta\cos\phi \ , \qquad\qquad x_2 = r\cos\theta\sin\phi \ ,$$
$$x_3 = r\sin\theta\cos\tau \ , \qquad\qquad x_4 = r\sin\theta\sin\tau \ , \tag{2.26}$$

and the relevant harmonic functions are now given by

$$H_1 = \frac{1}{\sqrt{(r^2+1)^2 - 4r^2\sin^2\theta}} \ , \qquad H_5 = \frac{1}{r^2} \ . \tag{2.27}$$

Performing the same expansion as before yields the leading deformation corresponding to this background with respect to the unperturbed system where both sets of branes lie at a point. The deformation contains only $SL(2,\mathbb{R})$ currents and reads

$$\cos 2\theta \ \partial x^+\bar{\partial}x^- = \Phi^{su}_{1;0,0}\Phi^{sl}_{1;-2,-2} K^+\bar{K}^+ \ . \tag{2.28}$$

Notice that this differs just by an overall sign from the $SL(2,\mathbb{R})$ deformation that appeared in the previous example. Therefore, if we put both types of branes on circles of the same size, the total deforming operator will be $\Phi^{sl}_{0;-1,-1}J^3\bar{J}^3$. As we already pointed out, this is analogous to the $\mathbb{R}_\phi \times SU(2)$ operator that deforms the system of NS5-branes from a point into a small circle [9]. Here, we see that this deformation treats both NS5- and F1-branes on an equal footing since it corresponds to putting on a circle *of the same radius* both types of branes simultaneously.

2.5 Elliptical deformations

We have seen so far that the SU(2) primary $\Phi^{su}_{1;m,\bar{m}}$ has appeared with $m = \bar{m} = 0$. The reason for that is that a circle deformation of either type of branes preserves the SO(2) symmetry associated with the plane where the deformation takes place. Hence, we expect that a generic planar deformation will break this isometry and trigger SU(2) primaries with $m, \bar{m} \neq 0$. To be concrete, let us consider a small elliptical deformation of the F1-branes, as described by

$$(x^1)^2 + (x^2)^2 = \epsilon^2\cos^2\psi \ . \tag{2.29}$$

The corresponding deformation of the harmonic function H_1 away from its point-like limit is, to leading order

$$\delta H_1 \sim \frac{1}{r^4}(\cos 2\theta + \cos^2\theta \cos 2\phi) \quad \Longrightarrow \quad \delta H_1^{-1} \sim \cos 2\theta + \cos^2\theta \cos 2\phi \ . \qquad (2.30)$$

This can be written in terms of WZW primaries and currents as

$$(\cos 2\theta + \cos^2\theta \cos 2\phi) \ \partial x^+ \bar\partial x^- \sim (\Phi^{su}_{1;0,0} + \Phi^{su}_{1;1,1} + \Phi^{su}_{1;-1,-1})\Phi^{sl}_{1;-2,-2} \ K^+ \bar K^+ \ . \qquad (2.31)$$

The first term gives rise to a perturbation that is the same as (2.28) corresponding to the circular deformation of the F1-branes. The other-two terms describe precisely the breaking of the U(1) symmetry due to the elliptical deformation and come on equal footing to ensure the reality of the perturbation.

3 Supersymmetric deformations of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$ theory

We will now proceed with a systematic scan of all operators that can trigger supersymmetric deformations of the original theory. For this purpose, we use worldsheet CFT techniques. As already mentioned in the introduction, spacetime supersymmetry does not require in the present framework $\mathcal{N} = 2$ superconformal invariance, which turns out to be preserved only for a subset of the deformations.

3.1 Spacetime supersymmetry of $\mathrm{AdS}_3 \times S^3$ background

All the configurations of the previous section preserve at least 1/4 of supersymmetry and therefore, if embedded in type II superstrings, they should maintain 8 supersymmetries. The special system where both types of branes are at the same point exhibits actually supersymmetry enhancement and preserves 16 supersymmetries. This matches the number of spacetime supercharges constructed in the $\mathrm{AdS}_3 \times S^3$ σ model, as we will review shortly following [12]. Subsequently, we would like to establish that the deformations we found in the previous section preserve 8 supercharges, in accordance with the analysis of the Killing spinor equations that we have performed there.

It is standard lore in string theory that spacetime supersymmetry is tied to the existence of extended worldsheet supersymmetry. However, the fact that we deal here with a curved timelike background, due to the AdS_3 factor in the metric, invalidates the usual argument due to [11], which refers to a Minkowski spacetime, and one has to follow a different procedure. The approach of [12] was to construct the spacetime supercharges directly, i.e. without employing the underlying $\mathcal{N} = 2$ supeconformal symmetry, and explicitly verify their BRST invariance. We will proceed here in a similar fashion and further discuss the appearance and the role of an $\mathcal{N} = 2$ superconformal algebra in section 3.2.

In order to construct the spacetime supercharges we should first bosonize the fermions of the theory. The fermions $\psi^\pm$, $\chi^\pm$ and χ^3, ψ^3 are bosonized in terms of three canonically normalized scalars fields. In order to capture all fermions, including the partners of the $\mathbb{T}^4$, we will introduce five bosons $H_i, i = 1, \ldots, 5$, obeying

$$H_i(z)H_j(w) = -\delta_{ij}\log(z-w) , i,j = 1,2,\ldots,5 \ . \qquad (3.1)$$

Recall that for a scalar field with canonical normalization we have

$$e^{iaH(z)}e^{ibH(w)} = (z-w)^{ab}e^{i[aH(z)+bH(w)]}\,, \tag{3.2}$$

where normal ordering is implied. Then, we have

$$\psi^\pm = e^{\pm iH_1}\,, \qquad \chi^\pm = e^{\pm iH_2}, \qquad \psi^3 = \frac{e^{iH_3} + e^{-iH_3}}{\sqrt{2}}\,, \qquad \chi^3 = \frac{e^{iH_3} - e^{-iH_3}}{\sqrt{2}}\,. \tag{3.3}$$

Correspondingly, we have the currents

$$\psi^+\psi^- = i\partial H_1\,, \quad \chi^+\chi^- = i\partial H_2\,, \quad \psi^3\chi^3 = -i\partial H_3\,. \tag{3.4}$$

The expression for χ^3 reflects the fact that its norm is negative. The fermions $\lambda^a, a = 1,\dots,4$ of the $\mathbb{T}^4$ are bosonized in a standard fashion as

$$\hat{\lambda}^\pm := \frac{1}{\sqrt{2}}(\lambda^1 \pm i\lambda^2) = e^{\pm iH_4}\,, \qquad \tilde{\lambda}^\pm := \frac{1}{\sqrt{2}}(\lambda^3 \pm i\lambda^4) = e^{\pm iH_5}\,. \tag{3.5}$$

Notice that $H_{1,2,4,5}^\dagger = H_{1,2,4,5}$ while $H_3^\dagger = -H_3$.

The supercharges take the usual form

$$Q = \oint dz\, e^{-\frac{\varphi}{2}} e^{\frac{i}{2}\sum_{i=1}^5 \epsilon_i H_i}\,, \tag{3.6}$$

with ϵ_i being ± 1 and φ being the bosonized superghost. The allowed values of ϵ_i are constrained due to the requirement of mutual-locality, which demands

$$\prod_{i=1}^5 \epsilon_i = 1\,, \tag{3.7}$$

and BRST invariance, which further dictates

$$\prod_{i=1}^3 \epsilon_i = -1\,. \tag{3.8}$$

It is fairly straightforward to see why the first condition is necessary.

The second condition comes out as follows. The BRST charge contains a term $Q_{\mathrm{BRST}} = \cdots + \gamma G^1 + \cdots$, where γ is one of the superghosts and the $\mathcal{N} = 1$ supercurrent $G^1 = \frac{1}{\sqrt{2}}(G^+ + G^-)$ contains the cubic terms

$$G^1_{3-\mathrm{Fermi}} = \psi^+\psi^-\psi^3 - \chi^+\chi^-\chi^3 \sim (\partial H_1 - \partial H_2)e^{iH_3} + (\partial H_1 + \partial H_2)e^{-iH_3}\,, \tag{3.9}$$

as can be found from the realization (C.2). These terms can give poles of order $\mathcal{O}(z^{-3/2})$ and $\mathcal{O}(z^{-1/2})$ in their OPE with the supercharges. Since the OPE of the superghost γ with $e^{-\frac{\varphi}{2}}$ is of order $\mathcal{O}(z^{1/2})$, the only potential problem comes from the $\mathcal{O}(z^{-3/2})$ poles, which therefore should cancel out. Explicitly, we find that the OPE

$$G^1_{3-\mathrm{Fermi}}(z)\ e^{\frac{i}{2}[\epsilon_1 H_1(w) + \epsilon_2 H_2(w) + \epsilon_3 H_3(w)]}\,, \tag{3.10}$$

is proportional to

$$\frac{\epsilon_1 - \epsilon_2}{z - w}(z - w)^{\frac{\epsilon_3}{2}} + \frac{\epsilon_1 + \epsilon_2}{z - w}(z - w)^{-\frac{\epsilon_3}{2}} \ . \tag{3.11}$$

Therefore, if $\epsilon_3 = 1$ we need $\epsilon_1 + \epsilon_2 = 0$ to cancel the $\mathcal{O}(z^{-3/2})$ pole from the second term and vice versa if $\epsilon_3 = -1$, i.e. we obtain condition (3.8).

Summarizing, the allowed supercharges are

$$\begin{aligned}
Q_{1\pm} &= e^{\frac{i}{2}[-H_1 - H_2 - H_3 \pm (H_4 - H_5)]} \ , \\
Q_{2\pm} &= e^{\frac{i}{2}[-H_1 + H_2 + H_3 \pm (H_4 - H_5)]} \ , \\
Q_{3\pm} &= e^{\frac{i}{2}[H_1 + H_2 - H_3 \pm (H_4 - H_5)]} \ , \\
Q_{4\pm} &= e^{\frac{i}{2}[H_1 - H_2 + H_3 \pm (H_4 - H_5)]} \ .
\end{aligned} \tag{3.12}$$

These are 8 supercharges and along with the contribution from the antiholomorphic sector we obtain in total 16 supercharges, which matches the number of supersymmetries preserved by the dual brane system.

3.2 Chiral primaries

Before proceeding with the analysis of the various supersymmetric deformations and preserved spacetime supercharges (section 3.3), we would like to pause and discuss the advertized superconformal symmetry.

In a theory with $\mathcal{N} = 2$ superconformal symmetry one can obtain a class of worldsheet supersymmetry-preserving marginal deformations by considering the chiral (and antichiral) primary operators. Since, however, for the backgrounds of interest the existence of spacetime supersymmetry is not tied to the $\mathcal{N} = 2$ worldsheet supersymmetry, one should not restrict to chiral primaries. As we will see below, the deformations originating from chiral primaries are indeed a small subset of the class of deformations preserving spacetime supersymmetry.

The reader might be puzzled by the above statement, referring to an $\mathcal{N} = 2$ superconformal algebra, which is not expected to be realized in Lorentzian backgrounds. In the σ model under consideration, however, a non-hermitian realization of such an algebra is available and displayed in appendix C. It can be understood as follows: The non-trivial part of the worldsheet theory is the factor $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)$, which can be further decomposed as $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \mathrm{U}(1) \times \mathbb{R}$. The coset factor $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$ provides a genuine $\mathcal{N} = 2$ (even $\mathcal{N} = 4$) superconformal algebra — the one present e.g. in the circular NS5-brane distribution — whereas the lightcone factor $\mathrm{U}(1) \times \mathbb{R}$ is presumably responsible for the lack of hermiticity. For our purposes, it is obviously natural to use the primaries and currents of the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ WZW models. Any further reference to the $\mathcal{N} = 2$ algebra should be understood in those terms.

Returning to our analysis we would like to use the chiral primary operators as supersymmetric seeds for marginal deformations so that we will focus on those that have conformal dimension $h = 1/2$. Subsequently, their R-charge should be $Q = \pm 1$. A quite

broad class of operators with $h = 1/2$ has the following form[4]

$$\Phi^{su}_{j;m}\Phi^{sl}_{j;m'}\mathcal{Y}\,, \tag{3.13}$$

where $\Phi^{su}_{j;m}, \Phi^{sl}_{j;m'}$ are affine primaries of the bosonic subalgebra of the full affine algebra of the super-WZW models and $\mathcal{Y}$ is any of the fermions of the theory. In this section we will actually restrict our analysis to the case where $\mathcal{Y}$ is a fermion in the SU(2) or the SL(2, $\mathbb{R}$) WZW models, since these operators are most relevant for the applications we have in mind, and we will consider the additional case where $\mathcal{Y}$ is a fermion from $\mathbb{T}^4$ in the next section. As usual we have suppressed the antiholomorphic indices in order to avoid unnecessary cluttering of the formulas. We hasten to add that for non-unitary CFTs the relation $h = \frac{Q}{2}$ is a necessary but not a sufficient condition for an operator to be chiral primary. Therefore, we can use it to restrict the possibilities, but we should still check explicitly if the operators we obtain are actual chiral primaries.

We start by noticing that under the U(1) R-current (C.3), $\psi^3 \pm \chi^3$ have charges $Q = \mp 1$. Instead, the other fermions have also a contribution from the fermionic part inside J^3_{T} or K^3_{T}. Therefore, if we use these fermions we should appropriately adjust m and m' in order to have vanishing $J^3_{\mathrm{T}} + K^3_{\mathrm{T}}$ charge and just obtain $Q = \pm 1$ from the other fermionic terms. The same is true of course when the fermion is $\psi^3 \pm \chi^3$, where we should ensure that $m + m' = 0$.

Therefore, we conclude that we have the following three classes of potential chiral primary operators

$$
\begin{aligned}
\Phi^{su}_{j;m}\Phi^{sl}_{j;m'}\chi^+\,, & \qquad m + m' + 1 = 0\,, \\
\Phi^{su}_{j;m}\Phi^{sl}_{j;m'}\psi^-\,, & \qquad m + m' - 1 = 0\,, \\
\Phi^{su}_{j;m}\Phi^{sl}_{j;m'}(\psi^3 - \chi^3)\,, & \qquad m + m' = 0\,.
\end{aligned}
\tag{3.14}
$$

Similarly we have a complementary set of potential antichiral operators with the appropriate fermions. So far these results do not depend on the particular values of m and m' or on the branch we choose for the SL(2, $\mathbb{R}$) primary. However, checking explicitly the chirality of these operators by computing their OPEs with G^+, reveals that, like the situation encountered in [9], only for specific charges m and m' and specific branches these operators are actually chiral primary.

From the first two classes we find that only $\Phi^{su}_{j;j}\Phi^{sl}_{j;-j-1}\chi^+$ and $\Phi^{su}_{j;-j}\Phi^{sl}_{j;j+1}\psi^-$ are chiral primary. It is worth noticing that, had we considered the non-normalizable primary of SL(2, $\mathbb{R}$) we would have found that the operator fails to be either primary or chiral. The purely bosonic pieces of the corresponding deformations are $K^+\Phi^{su}_{j;j}\Phi^{sl}_{j;-j-1}$ and

---

[4] We will focus mostly on normalizable operators in the SL(2, $\mathbb{R}$) model, since these correspond to deformations of the brane system. In other words by $\Phi^{sl}_{j;m}$ we mean the normalizable version of the operator with conformal weight $\Delta = -j(j+1)/k$. Recall that to each such conformal weight in SL(2, $\mathbb{R}$) there are associated two values of j related by reflection $j \leftrightarrow -j - 1$. The two values correspond to the normalizable and non-normalizable branch of the corresponding operator. For instance, the non-normalizable identity operator with $\Delta = 0$ has $j = -1, m = 0$ and is annihilated by all SL(2, $\mathbb{R}$) currents, in other words $\Phi^{sl}_{-1;0} \equiv 1$. This is the analogue of 1 in the linear dilaton theory. Instead, the operator with $j = 0$ is its normalizable version and the edge states of the two discrete representations with $m = \pm 1$ correspond to $e^{-q\phi}$ in the linear dilaton theory (see also comment in footnote 3).

$J^{-}\Phi^{su}_{j;-j}\Phi^{sl}_{j;j+1}$ respectively and they have vanishing R-charge as expected. Obviously a similar story holds for the antichiral operators which read $\Phi^{su}_{j;-j}\Phi^{sl}_{j;j+1}\chi^{-}$ and $\Phi^{su}_{j;j}\Phi^{sl}_{j;-j-1}\psi^{+}$, and which give rise to the deformations $K^{-}\Phi^{su}_{j;-j}\Phi^{sl}_{j;j+1}$ and $J^{+}\Phi^{su}_{j;j}\Phi^{sl}_{j;-j-1}$. There are no normal-ordering ambiguities since the primaries of the SU(2) and SL(2, $\mathbb{R}$) WZW theories that appear correspond to edge states of the spin j representations and are annihilated by the associated, with the perturbation current, operators. From the third class only the operator $\psi^{3} - \chi^{3}$ is chiral primary and leads to the deformation $J^{3} - K^{3}$. Notice that this last operator is actually non-normalizable in SL(2, $\mathbb{R}$).

To summarize, the chiral primaries of the theory are

$$\Phi^{su}_{j;j}\Phi^{sl}_{j;-j-1}\chi^{+}, \qquad \Phi^{su}_{j;-j}\Phi^{sl}_{j;j+1}\psi^{-}, \qquad \psi^{3} - \chi^{3} \tag{3.15}$$

and similarly the antichiral primaries are

$$\Phi^{su}_{j;-j}\Phi^{sl}_{j;j+1}\chi^{-}, \qquad \Phi^{su}_{j;j}\Phi^{sl}_{j;-j-1}\psi^{+}, \qquad \psi^{3} + \chi^{3} \ . \tag{3.16}$$

3.3 Spacetime supersymmetric deformations

We note that some of the deforming operators uncovered in the section 2 do not originate from the chiral primaries found above. For instance, consider $\Phi^{su}_{1;0,0}\Phi^{sl}_{1;-2,-2} K^{+}\bar{K}^{+}$ in (2.25) coming from the seed operator $\Phi^{su}_{1;0,0}\Phi^{sl}_{1;-2,-2} \chi^{+}\bar{\chi}^{+}$. The latter does not have the proper SU(2) charge to be a chiral primary. Since, however, the deformations arising from chiral primaries are guaranteed to preserve only the $\mathcal{N} = 2$ worldsheet supersymmetry, but not spacetime supersymmetry and, in any case, the $\mathcal{N} = 2$ does not seem to be tied to the existence of spacetime supersymmetry, we should check directly how many of the original supercharges are conserved by a very general class of deformations. Our findings will be in full consistency with the results of the section 2, which were based on supergravity.

Fermions in the SL(2, $\mathbb{R}$): we will consider a general ansatz for a seed operator of the type studied in [10], with form

$$A\Phi^{su}_{j;n}\Phi^{sl}_{j;m+1}\chi^{-} + B\Phi^{su}_{j;n}\Phi^{sl}_{j;m}\chi^{3} + C\Phi^{su}_{j;n}\Phi^{sl}_{j;m-1}\chi^{+} \ . \tag{3.17}$$

Notice that we will restrict ourselves only to NS sector operators. For certain values of A, B, C, corresponding to Clebsch-Gordan coefficients, this operator belongs to an irreducible representation with spin $j + 1$ of the SL(2, $\mathbb{R}$) generated by the total currents K^{i}_{T}. Acting on it with the $\mathcal{N} = 1$ supercurrent G^{1} and collecting the residues of the first order pole yields the actual deformation. The latter consists of the purely bosonic piece

$$A\Phi^{su}_{j;n}\Phi^{sl}_{j;m+1}K^{-} + \sqrt{2}B\Phi^{su}_{j;n}\Phi^{sl}_{j;m}K^{3} + C\Phi^{su}_{j;n}\Phi^{sl}_{j;m-1}K^{+} \ , \tag{3.18}$$

as well fermion bilinears.

It is obvious that all spacetime supercharges (3.12) commute with the purely bosonic piece of the deformation and potential obstructions result from the fermion bilinear pieces.

Grouping the latter according to the bosonic primaries they contain, since different primaries do not interfere with each other, we have:

$$(j+n)C\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m-1}\psi^+\chi^+\,,$$
$$\Phi^{su}_{j;n}\Phi^{sl}_{j;m-1}\Big((B(-1-j+m)+\sqrt2 Cm)\chi^+\chi^3+\sqrt2 Cn\psi^3\chi^+\Big)\,,$$
$$(j-n)C\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m-1}\psi^-\chi^+\,,$$
$$(j+n)B\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m}\psi^+\chi^3\,,$$
$$\Phi^{su}_{j;n}\Phi^{sl}_{j;m}\Big((\sqrt2 B-A(j-m)-C(j+m))\chi^+\chi^-+\sqrt2 Bn\psi^3\chi^3\Big)\,, \qquad (3.19)$$
$$(j-n)B\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m}\psi^-\chi^3\,,$$
$$(j+n)A\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m+1}\psi^+\chi^-\,,$$
$$\Phi^{su}_{j;n}\Phi^{sl}_{j;m+1}\Big((B(1+j+m)+\sqrt2 Am)\chi^-\chi^3+\sqrt2 An\psi^3\chi^-\Big)\,,$$
$$(j-n)A\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m+1}\psi^-\chi^-\,.$$

The term in the 5th line is a current and its action on any supercharge has always a pole since all supercharges contain H_2 and H_3. The condition it leads to is

$$\big(\sqrt2 B-A(j-m)-C(j+m)\big)\epsilon_2-\sqrt2 Bn\epsilon_3=0\,. \qquad (3.20)$$

In total we have 15 fermion bilinears. We present in the table below the result of the action of the fermion bilinears on the supercharges[5] where a tick means that the supercharge commutes with the bilinear. We have excluded the current terms $\psi^+\psi^-$, $\chi^+\chi^-$ and $\psi^3\chi^3$ since they do not commute with any supercharge.

	$\psi^+\chi^+$	$\chi^+(\psi^3,\chi^3)$	$\psi^-\chi^+$	$\psi^+(\psi^3,\chi^3)$	$\psi^-(\psi^3,\chi^3)$	$\psi^+\chi^-$	$\chi^-(\psi^3,\chi^3)$	$\psi^-\chi^-$
Q_1	Q_3	Q_2	✓	Q_3	✓	✓	✓	✓
Q_2	✓	✓	✓	$(Q_3,-Q_3)$	✓	Q_4	$(Q_1,-Q_1)$	✓
Q_3	✓	✓	✓	✓	Q_2	✓	Q_4	Q_1
Q_4	✓	$(Q_3,-Q_3)$	Q_2	✓	$(Q_1,-Q_1)$	✓	✓	✓

Let us now analyze the conditions for preserving at least 4 supercharges, for instance $Q_{2\pm}$ and $Q_{3\pm}$. Then we get the six conditions

$$(j\pm n)A=0\,,\quad (j\pm n)B=0\,,\quad (1+j+m)B+\sqrt2 Am=0\,,\quad nA=0\,, \qquad (3.21)$$

plus two more from (3.20) corresponding to $Q_{2\pm}$ (with $\epsilon_2=\epsilon_3=1$) and $Q_{3\pm}$ (with $\epsilon_2=-\epsilon_3=1$). Except for the case $j=n=0$ the only solution is $A=B=0$. From the current condition $C(j+m)=0$ and since $C\neq0$ with obtain eventually $m=-j$. Therefore the seed operator that leads to a deformation preserving 4 supercharges, from the holomorphic sector, is $\Phi^{su}_{j;n}\Phi^{sl}_{j;-j-1}\chi^+$. Similarly the operator $\Phi^{su}_{j;n}\Phi^{sl}_{j;j+1}\chi^-$ preserves the complementary set of supercharges $Q_{1\pm}$ and $Q_{4\pm}$. Furthermore, it is straightforward

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[5]In the table we suppress the indices $\pm$ from the supercharges to avoid cluttering.

to check that there are no other combinations of supercharges that can be preserved except for the two ones above.

If $j = n = 0$ we find that the supercharges $Q_{2\pm}$ and $Q_{3\pm}$ are preserved provided that

$$\begin{aligned}
B(1 + m) + \sqrt{2}Am &= 0 \,, \\
\sqrt{2}B + m(A - C) &= 0 \,.
\end{aligned} \tag{3.22}$$

However these two conditions (along with $n = 0$) imply that the 2nd term in (3.19) has vanishing coefficient and therefore $Q_{1\pm}$ and $Q_{4\pm}$ are also preserved! If $m \neq 0$ the general solution of that system yields the deforming operator (up to an overall multiplicative constant)

$$(m + 1)\Phi^{sl}_{0;m+1}K^- - 2m\Phi^{sl}_{0;m}K^3 + (m - 1)\Phi^{sl}_{0;m-1}K^+ \,, \tag{3.23}$$

while for $m = 0$ we have the deforming operator

$$A\Phi^{sl}_{0;1}K^- + C\Phi^{sl}_{0;-1}K^+ \,. \tag{3.24}$$

Fermions in the SU(2): let us consider now operators of the form

$$A\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m}\psi^- + B\Phi^{su}_{j;n}\Phi^{sl}_{j;m}\psi^3 + C\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m}\psi^+ \,. \tag{3.25}$$

The purely bosonic piece of the deformation induced by this operator reads

$$A\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m}J^- + \sqrt{2}B\Phi^{su}_{j;n}\Phi^{sl}_{j;m}J^3 + C\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m}J^+ \,. \tag{3.26}$$

The fermion bilinear terms are grouped again according to the bosonic primaries as follows:

$$\begin{aligned}
&(j + 1 - m)C\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m-1}\psi^+\chi^+ \,, \\
&(j + 1 - m)B\Phi^{su}_{j;n}\Phi^{sl}_{j;m-1}\psi^3\chi^+ \,, \\
&(j + 1 - m)A\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m-1}\psi^-\chi^+ \,, \\
&\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m}\left(\sqrt{2}Cm\psi^+\chi^3 + \left(\sqrt{2}C + B(j - n)\right)\psi^+\psi^3\right) \,, \\
&\Phi^{su}_{j;n}\Phi^{sl}_{j;m}\left(\left(\sqrt{2}B + A(1 + j + n) - C(1 + j - n)\right)\psi^+\psi^- + \sqrt{2}Bm\psi^3\chi^3\right) \,, \quad (3.27) \\
&\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m}\left(\sqrt{2}Am\psi^-\chi^3 + \left(B(j - n) - \sqrt{2}An\right)\psi^-\psi^3\right) \,, \\
&(j + 1 + m)C\Phi^{su}_{j;n-1}\Phi^{sl}_{j;m+1}\psi^+\chi^- \,, \\
&(j + 1 + m)B\Phi^{su}_{j;n}\Phi^{sl}_{j;m+1}\psi^3\chi^- \,, \\
&(j + 1 + m)A\Phi^{su}_{j;n+1}\Phi^{sl}_{j;m+1}\psi^-\chi^- \,.
\end{aligned}$$

It is straightforward to check that $\Phi^{su}_{j;j}\Phi^{sl}_{j;m}\psi^+$ and $\Phi^{su}_{j;-j}\Phi^{sl}_{j;m}\psi^-$ preserve half of the original supercharges, that is $(Q_{3\pm}, Q_{4\pm})$ and $(Q_{1\pm}, Q_{2\pm})$, respectively. Furthermore, the operators $\Phi^{sl}_{0;\pm 1}\psi^3$ preserve the supercharges $Q_{1,\pm}, Q_{4\pm}$ for $m = 1$ and $Q_{2,\pm}, Q_{3\pm}$ for $m = -1$. No other combinations of supercharges can be preserved. We should emphasize that in the general analysis above we take the SL(2, $\mathbb{R}$) primaries in the normalizable branch.

All these operators contain the chiral/antichiral primaries found previously but in general there are by far more 1/2-BPS operators. It can be checked that these extra operators

lead to deformations that do not preserve the original $\mathcal{N} = 2$ SCFT symmetry.[6] As a rule, the charge of the WZW primary that comes from the same model as the fermion is fixed while the other primary has arbitrary charge. We should also mention that although the marginal deformations originating from chiral primaries can be argued to be exactly marginal, this is not possible for the deformations coming from the above operators (although that does not necessarily imply that these deformations are not exactly marginal). Note also that our operators do not match, in general, the operators of [10], where A, B, C are fixed in terms of Clebsch-Gordan coefficients, except when one of the states is at the boundary of the representation space and two out of the three Clebsch-Gordan coefficients vanish.

A mixed operator: let us finally check the operator $\Phi^{sl}_{j;m}(\psi^3 - \chi^3)$. It is chiral primary for $j = -1, m = 0$ (when $\Phi^{sl}_{-1;0} \equiv 1$) and then corresponds to a non-normalizable deformation. It makes also sense as a seed operator if $j = 0$ and $m = -1$ so that it gives rise to a marginal normalizable deformation. The bosonic piece of this deformation reads

$$\Phi^{sl}_{j;m}(J^3 - K^3) \,, \tag{3.28}$$

while the fermionic one is given by the sum of the following terms

$$
\begin{aligned}
&(1 + j - m)\Phi^{sl}_{j;m-1}(\psi^3\chi^+ + \chi^+\chi^3) \,, \\
&\Phi^{sl}_{j;m}(\chi^+\chi^- - m\psi^3\chi^3 - \psi^+\psi^-) \,, \\
&(1 + j + m)\Phi^{sl}_{j;m+1}(\psi^3\chi^- + \chi^-\chi^3) \,.
\end{aligned}
\tag{3.29}
$$

The term in the second line implies that $\epsilon_2 + m\epsilon_3 - \epsilon_1 = 0$, which is not possible to satisfy for $m = -1$. For $m = 0$, this condition becomes $\epsilon_1 = \epsilon_2$ which is satisfied only for the supercharges $Q_{1\pm}$ and $Q_{3\pm}$. Then, using also $j = -1$ we see that the terms in the first and third lines in (3.30) are vanishing as well. Hence, we have a 1/2 BPS deformation, which however is non-normalizable. One could further consider more general combinations of operators with fermions from both WZW models, however it turns out that they do not lead to other supersymmetric operators besides the one we found above.

Summary: to summarize, we have found the following classes of seed operators that yield 1/2 BPS deformations in spacetime:

$$\Phi^{su}_{j;n}\Phi^{sl}_{j;\mp j \mp 1}\chi^{\pm} \,, \quad j \neq 0 \,, \qquad \Phi^{su}_{j;\pm j}\Phi^{sl}_{j;m}\psi^{\pm} \,, \qquad \Phi^{sl}_{0;\pm 1}\psi^3 \,, \qquad \psi^3 - \chi^3 \,. \tag{3.30}$$

In addition, the following operators yields deformations that do not break any supersymmetry:

$$(m+1)\Phi^{sl}_{0;m+1}\chi^- - \sqrt{2}m\Phi^{sl}_{0;m}\chi^3 + (m-1)\Phi^{sl}_{0;m-1}\chi^+ \,, \quad m \neq 0 \tag{3.31}$$

and

$$A\Phi^{sl}_{0;1}\chi^- + C\Phi^{sl}_{0;-1}\chi^+ \,. \tag{3.32}$$

Finally, let us mention that none of the operators we have studied so far can preserve *only* 1/4 of the original supersymmetry.

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[6] Notice that in order to have an $\mathcal{N} = 2$ preserving deformation it is sufficient but not necessary that the seed operator is chiral or antichiral primary.

3.4 Brane interpretation and comments

All geometric deformations of the pointlike brane system are captured by the ansatz (2.1) with the functions H_1 and H_5 depending on the common transverse space, i.e. on the radial coordinate ρ as well as on the SU(2) coordinates. Therefore, from the set of spacetime supersymmetric operators we uncovered in the previous subsection only a subclass can be given an interpretation in terms of a deformed brane system. This is the subclass whose SL(2, $\mathbb{R}$) primary depends only on ρ and which involves only the $K^+, \bar{K}^+$ currents. Otherwise it is easy to see, using the formulas from the appendix B, that the deformation will depend also on the coordinates $x^\pm$, therefore loosing its brane description. Hence, *the operators that could a priori* correspond to geometric deformations of the brane system are

$$\Phi^{su}_{j;n,\bar{n}}\Phi^{sl}_{j;-j-1,-j-1}\chi^+\bar{\chi}^+\,, j\neq 0\,, \qquad \Phi^{su}_{j;\pm j,\pm j}\Phi^{sl}_{j;-j-1,-j-1}\psi^\pm\bar{\psi}^\pm\,, \qquad \Phi^{sl}_{0;-1,-1}\psi^3\bar{\psi}^3\,, \tag{3.33}$$

where we reinstalled the anti-holomorphic indices for concreteness.

Notice that the only maximally supersymmetric operator that has the right form to yield a brane deformation is $\Phi^{sl}_{0;-1,-1}\chi^+\bar{\chi}^+ \sim e^{2\rho}\partial x^+\bar{\partial}x^-$ and therefore it trivially corresponds to an overall rescaling of the coordinates $x^\pm$. This is consistent as there are no deformations of the original F1-NS5-brane system that preserve its total supersymmetry. The rest of the operators that preserve the full supersymmetry correspond to diffeomorphisms of the AdS$_3$ metric, as can be verified by computing the scalar curvature of the deformed metric, and therefore they have a trivial physical effect.

We see now that $\Phi^{sl}_{0;-1,-1}J^3\bar{J}^3$ and $\Phi^{su}_{1;0,0}\Phi^{sl}_{1;-2,-2}\,K^+\bar{K}^+$, which appear when we put the branes on circles, are accounted for by the third and first operators of the above list, respectively. It is also important that these two classes of operators preserve the same set of supercharges, that is $Q_{2\pm}$ and $Q_{3\pm}$, so that the combined deformation is still supersymmetric as it should. The same is true for the operators $\Phi^{su}_{1;1,1}\Phi^{sl}_{1;-2,-2}\,K^+\bar{K}^+$ and $\Phi^{su}_{1;-1,-1}\Phi^{sl}_{1;-2,-2}\,K^+\bar{K}^+$ that describe an elliptical deformation of the F1-branes. We notice now that from the supergravity point of view all deformations preserve the same set of supercharges, since the form of the Killing spinors is not related to the actual expressions for the harmonic functions H_1 and H_5, and therefore the operators that yield brane deformations should be only those commuting with the supercharges $Q_{2\pm}$ and $Q_{3\pm}$ that are preserved by the circular and elliptical deformation. Therefore out of (3.33) we should further restrict only to the operators

$$\Phi^{su}_{j;n,\bar{n}}\Phi^{sl}_{j;-j-1,-j-1}\chi^+\bar{\chi}^+\,, \quad j\neq 0\,, \qquad \Phi^{sl}_{0;-1,-1}\psi^3\bar{\psi}^3\,. \tag{3.34}$$

It is a bit surprising that $\Phi^{su}_{j;\pm j,\pm j}\Phi^{sl}_{j;-j-1,-j-1}\psi^\pm\bar{\psi}^\pm$ have to be excluded since similar operators in [9], containing linear dilaton vertex operators instead of SL(2, $\mathbb{R}$) primaries, where argued to account for the geometric deformations of the pointlike NS5-brane system (along with the analogue of $\Phi^{sl}_{0;-1,-1}\psi^3\bar{\psi}^3$ that describes the circular deformation). However, besides the fact that $\Phi^{su}_{j;\pm j,\pm j}\Phi^{sl}_{j;-j-1,-j-1}\psi^\pm\bar{\psi}^\pm$ do not preserve the same set of supercharges as $\Phi^{sl}_{0;-1,-1}\psi^3\bar{\psi}^3$, we cannot use them in any case to construct a real deformation that preserves supersymmetry.

The reason is that we cannot construct a real operator by using only the currents J^+ and $\bar{J}^+$ and, on the other hand, operators with ψ^+ and its complex conjugate ψ^- preserve complementary sets of supercharges, i.e. $(Q_{3\pm}, Q_{4\pm})$ and $(Q_{1\pm}, Q_{2\pm})$ respectively, as we have already seen. Therefore, we cannot construct a real supersymmetric deformation using these operators. Two observations are now in order. First, this problem does not arise when we use the $\mathrm{SL}(2,\mathbb{R})$ fermions $\chi^+, \bar{\chi}^+$ because the currents $K^+, \bar{K}^+$ combine by themselves to a real operator. Second, this issue did not also arise in the setup of [9] because the analogues of $\Phi^{su}_{j;\pm j,\pm j}\Phi^{sl}_{j;-j-1,-j-1}\psi^\pm\bar{\psi}^\pm$ preserve the full amount of supersymmetry (16 supercharges) of the original undeformed NS5-brane configuration.

We would like to close this section with a final remark related to the fact that the levels of both WZW models are identified with the number of NS5-branes N_5. As a consequence, the number of operators in the first expression in (3.33) — i.e. those who survive the reality condition and truly generate supersymmetric deformations — scales approximately as N_5^3. From the brane point of view we would have expected $4(N_1 + N_5)$ possible deformations, since we can move all branes arbitrarily. It is not clear to us how this discrepancy should be interpreted (and eventually fixed), since the weak-string-coupling regime that guaranties the validity of the CFT analysis demands $N_1 \gg N_5$ (see (2.13)), which sets no order between N_5^3 and $4(N_1 + N_5)$.

4 More operators in the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathbb{T}^4$ theory

A large class of operators that give rise to marginal deformations consists of bosonic primaries of the above models with vanishing total conformal weight along with a fermion, so that the overall conformal weight is $h = 1/2$. The simplest and most natural construction involves two primaries. Therefore we have the following two classes of operators. Either we use (3.13), that is

$$\text{First class}: \qquad \Phi^{sl}_{j;m}\Phi^{su}_{j;n}\mathcal{Y} \tag{4.1}$$

or

$$\text{Second class}: \qquad \Phi^{sl}_{j;m}e^{ip_a Y^a}\mathcal{Y}, \tag{4.2}$$

where $\mathcal{Y}$ denotes a fermion in one of the WZW models or in the 4-torus and $Y^a, a = 1, 2, 3, 4$ are free bosons describing the 4-torus. We will also use the complex combinations $\hat{Y}^\pm = Y^1 \pm iY^2$ and $\tilde{Y}^\pm = Y^3 \pm iY^4$ in the construction of the $\mathcal{N} = 2$ superconformal algebra in appendix C. The condition $\frac{j(j+1)}{k} - \frac{1}{2}\sum_a p_a^2 = 0$ should hold (we consider for simplicity only momentum modes on the 4-torus) so that the second class of operators have conformal weight $h = 1/2$. Notice that the momenta p_a are quantized since the coordinates Y^a are compact but j is an arbitrary real number in the range

$$-\frac{1}{2} \leqslant j \leqslant \frac{k-1}{2}. \tag{4.3}$$

4.1 First class

Operators in the first class with the fermion $\mathcal{Y}$ being either in the $\mathrm{SL}(2,\mathbb{R})$ or the $\mathrm{SU}(2)$ part of the theory were studied in the previous section and we saw that several of them

can lead to deformations that preserve one-half of the original spacetime supersymmetry. For this to happen the charge n or m from the WZW model, where $\mathcal{Y}$ belongs, has to be fixed appropriately with respect to j. Furthermore, some of these operators correspond to geometric deformations of the F1-NS5-brane system similar to those studied in [7].

The operators (4.1) involving fermions from the 4-torus give rise to deformations of the moduli of the torus which, in general, will depend on the $SL(2,\mathbb{R})$ and $SU(2)$ coordinates through the corresponding affine primaries. We still have to check if any of those give rise to supersymmetric deformations but since the form of these deformations is not consistent with the general ansatz (2.1), we expect that none of those can preserve any supersymmetry. It is an interesting exercise to see how this happens.

Let us start from the seed operator

$$\Phi^{sl}_{j;m}\Phi^{su}_{j;n}\hat{\lambda}^+ \,, \tag{4.4}$$

where we consider a specific complex fermion from the 4-torus (obviously the analysis is similar for all other 4-torus fermions). The deforming operator then reads

$$\Phi^{sl}_{j;m}\Phi^{su}_{j;n}\partial\hat{Y}^+ \,, \tag{4.5}$$

The associated fermion bilinears and the supercharges that commute with them are

$$
\begin{aligned}
-(1+j-m)\Phi^{sl}_{j;m-1}\Phi^{su}_{j;n}\chi^+\hat{\lambda}^+ \,, \quad & Q_{1+}, \quad Q_{2\pm}, \quad Q_{3\pm}, \quad Q_{4+}, \\
(j+n)\Phi^{sl}_{j;m}\Phi^{su}_{j;n-1}\psi^+\hat{\lambda}^+ \,, \quad & Q_{1+}, \quad Q_{2+}, \quad Q_{3\pm}, \quad Q_{4\pm}, \\
-\sqrt{2}m\Phi^{sl}_{j;m}\Phi^{su}_{j;n}\chi^3\hat{\lambda}^+ \,, \quad & Q_{1+}, \quad Q_{2+}, \quad Q_{3+}, \quad Q_{4+}, \\
\sqrt{2}n\Phi^{sl}_{j;m}\Phi^{su}_{j;n}\psi^3\hat{\lambda}^+ \,, \quad & Q_{1+}, \quad Q_{2+}, \quad Q_{3+}, \quad Q_{4+}, \\
(j-n)\Phi^{sl}_{j;m}\Phi^{su}_{j;n+1}\psi^-\hat{\lambda}^+ \,, \quad & Q_{1\pm}, \quad Q_{2\pm}, \quad Q_{3+}, \quad Q_{4+}, \\
(1+j+m)\Phi^{sl}_{j;m+1}\Phi^{su}_{j;n}\chi^-\hat{\lambda}^+ \,, \quad & Q_{1\pm}, \quad Q_{2+}, \quad Q_{3+}, \quad Q_{4\pm} \,.
\end{aligned}
\tag{4.6}
$$

We notice that there is a common set of 4 commuting supercharges $Q_{i+}, i = 1,\ldots,4$ and that therefore these operators preserve $1/2$ of the original supersymmetry. However, in order to get a real deformation we should add the complex conjugates of the above deforming operators, which, as it can be easily seen, preserve the complementary set $Q_{i-}, i = 1,\ldots,4$. Therefore, there are no (real) supersymmetric deformations of this type, in accord with the fact that they are not expected from the supergravity analysis. It is also elementary to show that the usual moduli deformation of the torus, i.e. of the form $\partial\hat{Y}^+\bar{\partial}\hat{Y}^- \pm \partial\hat{Y}^-\bar{\partial}\hat{Y}^+$, commute with all supercharges as they should.

4.2 Second class

The second class of operators, which contain a primary from the 4-torus, leads also to $1/2$ BPS deformations when $\mathcal{Y}$ is a fermion from the $SL(2,\mathbb{R})$ model. In particular, the following operators yield deformations preserving 8 supercharges (including the holomorphic and antiholomorphic sectors): $\Phi^{sl}_{j;\mp j\mp 1}e^{ip_aY^a}\chi^\pm$. The corresponding deformations contain the null currents of the $SL(2,\mathbb{R})$ WZW model and hence they reflect a situation where the harmonic function H_1 in (2.1) depends on the coordinates y^a of the 4-torus.

Let us find out which supercharges can be preserved. We select as seed

$$\Phi_{j;m}^{sl}e^{ip_aY^a}\chi^+ \tag{4.7}$$

and we obtain the following 2-fermion terms in the associated deformation

$$a_b\Phi_{j;m}^{sl}e^{ip_aY^a}\lambda^b\chi^+ \,,$$
$$\sqrt{2}(1+m)\Phi_{j;m}^{sl}e^{ip_aY^a}\chi^+\chi^3 \,, \tag{4.8}$$
$$-(1+j+m)\Phi_{j;m}^{sl}e^{ip_aY^a}\chi^+\chi^- \,.$$

Since no supercharges commute with the third term we have to set $m = -1 - j$. The second term preserves $Q_{2\pm}, Q_{3\pm}$ and these are also preserved by the first term as well, due to the fact that they have $\epsilon_2 = 1$. Therefore we have a 1/2 BPS deformation. Similarly, such operators with χ^- preserve the complementary set $Q_{1\pm}$ and $Q_{4\pm}$.

Brane configurations corresponding to deformations of the F1-NS5-brane system driven by this type of operators were studied in [14], where it was shown that they are solutions of the equations of motion and preserve 1/4 of the original supersymmetry provided that the harmonic condition on H_1 changes to

$$\left(\partial_x^2 + H_5(x)\partial_y^2\right)H_1(x,y) = 0 \,. \tag{4.9}$$

A simple class of solutions of that equation, with H_5 being the standard near-horizon form of the harmonic function on the transverse space $H_5 = 1/r^2$, can be found by assuming a factorized form of $H_1(x,y) = f(x)g(y)$. We get two equations

$$r^2\frac{\partial_x^2 f(x)}{f(x)} = -\frac{\partial_y^2 g(y)}{g(y)} = c \,. \tag{4.10}$$

Since the coordinates y^b parametrize 4-torus, the solution of the second equation are of the form $g(y) = e^{ia_by^b}$ with the condition $c = \sum_b a_b^2$. Assuming furthermore that $f(x)$ depends only on the radial coordinate r yields

$$r^2 f''(r) + 3r f'(r) - cf(r) = 0 \,, \tag{4.11}$$

with solutions $f(r) = r^{-1\pm\sqrt{1+c}}$. This solution is a deformation of the original harmonic solution $H_1 = 1/r^2$. Recall that the latter corresponds to the F1-branes fully smeared on the 4-torus. The deformation reflects a situation where some momentum modes on the 4-torus are condensed and have to be compensated by a change of the profile of the F1-branes. This change of profile can be thought of as a deformation of the original smooth instanton to which the smeared F1-branes correspond to. Therefore, these deformations trigger infinitesimal motions in the instanton moduli space, the latter being the Higgs branch of the F1-NS5-system.

The conformal field theory description of these deformations is provided by the operators $\Phi_{j;-j-1}^{sl}e^{ip_aY^a}\chi^+$ which preserve the supercharges $Q_{2\pm}$ and $Q_{3\pm}$. These are exactly the supercharges preserved by the operators corresponding to the circular and elliptical deformation studied previously, in perfect agreement with the fact that the form of the Killing

spinors does not depend on the explicit form of the functions H_1 and H_5, even if we use the more general ansatz of [14]. The relation between the a_b and c is the classical analogue of the quantum relation between p_b and j that results from the condition of conformal invariance. We should mention that operators of this type have not been considered so far in discussions of the $\text{AdS}_3/\text{CFT}_2$ duality and it would be very interesting to elucidate their role in that context.

On the other side, it is not meaningful to give a y-dependence on the $H_5(x)$ harmonic function since it would imply a dependence of the harmonic function describing the NS5-branes on some of their worldvolume coordinates. Therefore, operators of the form (4.7) but with a fermion in the SU(2) WZW model should not yield exactly marginal deformations. Here we will restrict ourselves to showing that they cannot yield a real deformation that preserves supersymmetry.

Taking as seed the operator

$$\Phi^{sl}_{j;m} e^{ip_a Y^a} \psi^+ \, , \tag{4.12}$$

yields the following fermion bilinears

$$\begin{aligned}
&a_b \Phi^{sl}_{j;m} e^{ip_a Y^a} \lambda^b \psi^+ \, , \\
&(1 + j - m)\Phi^{sl}_{j;m-1} e^{ip_a Y^a} \psi^+ \chi^+ \, , \\
&\sqrt{2}(m-1)\Phi^{sl}_{j;m} e^{ip_a Y^a} \psi^+ \chi^3 \, , \\
&-\sqrt{2}(1 + j + m)\Phi^{sl}_{j;m+1} e^{ip_a Y^a} \psi^+ \chi^- \, .
\end{aligned} \tag{4.13}$$

All these terms preserve simultaneously the supercharges $Q_{3\pm}$ and $Q_{4\pm}$. However, as was the case with first class operators containing SU(2) fermions, in order to construct a real deformation we should also add the complex conjugate operator that involves the fermion ψ^- and these preserve the complementary set of supercharges $(Q_{1\pm}, Q_{2\pm})$, as it can easily be seen.

Finally, operators of the second class with a fermion from the 4-torus are excluded due to the same reason we excluded operators of the type (4.4).

Acknowledgments

A SU(2) conventions

We use the parametrization of the SU(2) matrix element employed in [7],

$$\begin{pmatrix} \tilde{g}_{++} & \tilde{g}_{+-} \\ \tilde{g}_{-+} & \tilde{g}_{--} \end{pmatrix} = \begin{pmatrix} \cos\theta \; e^{i\phi} & \sin\theta \; e^{i\tau} \\ -\sin\theta \; e^{-i\tau} & \cos\theta \; e^{-i\phi} \end{pmatrix} , \tag{A.1}$$

to obtain the following semiclassical expression for the primaries

$$\Phi^{su}_{j;j,j} = \tilde{g}^{2j}_{++} , \qquad \Phi^{su}_{j;-j,-j} = \tilde{g}^{2j}_{--} , \qquad \Phi^{su}_{j;j,-j} = \tilde{g}^{2j}_{+-} , \qquad \Phi^{su}_{j;-j,j} = \tilde{g}^{2j}_{-+} . \tag{A.2}$$

The SU(2) primary $\Phi^{su}_{j;m,\bar{m}}$ at level $k-2$ has conformal weight

$$\Delta = \frac{j(j+1)}{k} , \tag{A.3}$$

with j and m being half-integers in the ranges

$$0 \leqslant j \leqslant \frac{k-2}{2} , \quad -j \leqslant m \leqslant j . \tag{A.4}$$

The left- and right-moving currents of the theory are given by

$$\begin{aligned} J^1 &= 2\big(\sin(\phi+\tau)\partial\theta + \cos(\phi+\tau)\sin\theta\cos\theta(\partial\tau - \partial\phi) \big) , \\ J^2 &= 2\big(\cos(\phi+\tau)\partial\theta - \sin(\phi+\tau)\sin\theta\cos\theta(\partial\tau - \partial\phi) \big) , \\ J^3 &= 2\big(\cos^2\theta\partial\phi + \sin^2\theta\partial\tau \big) , \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} \bar{J}^1 &= -2\big(\sin(\phi-\tau)\bar{\partial}\theta + \cos(\phi-\tau)\sin\theta\cos\theta(\bar{\partial}\tau + \bar{\partial}\phi) \big) , \\ \bar{J}^2 &= 2\big(\cos(\phi-\tau)\bar{\partial}\theta + \sin(\phi-\tau)\sin\theta\cos\theta(\bar{\partial}\tau + \bar{\partial}\phi) \big) , \\ \bar{J}^3 &= 2\big(\cos^2\theta\bar{\partial}\phi - \sin^2\theta\bar{\partial}\tau \big) . \end{aligned} \tag{A.6}$$

The action of the SU(2) affine currents $J^3, J^\pm = J^1 \pm iJ^2$ on a primary field $\Phi^{su}_{j;m,\bar{m}}$ is given by the OPEs

$$\begin{aligned} J^3(z)\Phi^{su}_{j;m,\bar{m}}(w,\bar{w}) &= \frac{m}{z-w}\Phi^{su}_{j;m,\bar{m}}(w,\bar{w}) , \\ J^\pm(z)\Phi^{su}_{j;m,\bar{m}}(w,\bar{w}) &= \frac{j\mp m}{z-w}\Phi^{su}_{j;m\pm 1,\bar{m}}(w,\bar{w}) . \end{aligned} \tag{A.7}$$

The bosonic current algebra reads

$$\begin{aligned} J^3(z)J^3(w) &\sim \frac{k-2}{2}\frac{1}{(z-w)^2} , \\ J^3(z)J^\pm(w) &\sim \pm\frac{J^\pm(w)}{z-w} , \\ J^+(z)J^-(w) &\sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w} , \end{aligned} \tag{A.8}$$

at level $k-2$. The corresponding fermions satisfy

$$\begin{aligned} \psi^3(z)\psi^3(w) &\sim \frac{1}{z-w} , \\ \psi^+(z)\psi^-(w) &\sim \frac{1}{z-w} . \end{aligned} \tag{A.9}$$

B SL(2, ℝ) conventions

A two-dimensional matrix realization of the $SL(2, \mathbb{R})$ algebra is given in terms of Pauli matrices as follows:

$$k^1 = \frac{i}{2}\sigma^2\,, \qquad\qquad k^2 = -\frac{i}{2}\sigma^1\,, \qquad\qquad k^3 = \frac{1}{2}\sigma^3\,. \qquad\text{(B.1)}$$

These matrices satisfy the $SL(2, \mathbb{R})$ commutation relations

$$\left[k^1, k^2\right] = -ik^3\,, \qquad\qquad \left[k^2, k^3\right] = ik^1\,, \qquad\qquad \left[k^3, k^1\right] = ik^2\,, \qquad\text{(B.2)}$$

and therefore $k^{\pm} = k^1 \pm ik^2$ and k^3 satisfy

$$\left[k^3, k^+\right] = k^+\,, \qquad\qquad \left[k^3, k^-\right] = -k^-\,, \qquad\qquad \left[k^+, k^-\right] = -2k^3\,, \qquad\text{(B.3)}$$

which is the form of the $SL(2, \mathbb{R})$ we employ in the construction of the $\mathcal{N} = 2$ superconformal algebra.

We will parametrize the matrix element of $SL(2, \mathbb{R})$ as

$$g = \begin{pmatrix} e^{\rho} & e^{\rho}x^+ \\ e^{\rho}x^- & e^{-\rho} + x^+x^-e^{\rho} \end{pmatrix} = \begin{pmatrix} g_{++} & g_{+-} \\ g_{-+} & g_{--} \end{pmatrix}\,. \qquad\text{(B.4)}$$

The right-invariant 1-forms are $j_{\mathrm{R}}^a = -i\mathrm{tr}(dg g^{-1} k^a)$ and they read

$$\begin{aligned} j_{\mathrm{R}}^1 &= -\frac{i}{2}\Big(dx^- + 2x^- dr - e^{2\rho}\big(1 + (x^-)^2\big)dx^+\Big)\,, \\ j_{\mathrm{R}}^2 &= -\frac{1}{2}\Big(dx^- + 2x^- dr + e^{2\rho}\big(1 - (x^-)^2\big)dx^+\Big)\,, \\ j_{\mathrm{R}}^3 &= i(e^{2\rho}x^- dx^+ - dr)\,, \end{aligned} \qquad\text{(B.5)}$$

while the left-invariant 1-forms $j_{\mathrm{L}}^a = -i\mathrm{tr}(g^{-1}dg k^a)$ read

$$\begin{aligned} j_{\mathrm{L}}^1 &= \frac{i}{2}\Big(dx^+ + 2x^+ dr - e^{2\rho}\big(1 + (x^+)^2\big)dx^-\Big)\,, \\ j_{\mathrm{L}}^2 &= -\frac{1}{2}(dx^+ + 2x^+ dr + e^{2\rho}\big(1 - (x^+)^2\big)dx^-)\,, \\ j_{\mathrm{L}}^3 &= i(e^{2\rho}x^+ dx^- - dr)\,. \end{aligned} \qquad\text{(B.6)}$$

The Cartan-Killing metric $ds_{\mathrm{CK}}^2 = (j_{\mathrm{R}}^1)^2 + (j_{\mathrm{R}}^2)^2 - (j_{\mathrm{R}}^3)^2 = (j_{\mathrm{L}}^1)^2 + (j_{\mathrm{L}}^2)^2 - (j_{\mathrm{L}}^3)^2$ is

$$ds^2 = d\rho^2 + e^{2\rho}dx^+ dx^-\,. \qquad\text{(B.7)}$$

Consequently, the left- and right-moving currents of the WZW model are

$$\begin{aligned} K^1 &= -\frac{i}{2}\Big(\partial x^- + 2x^-\partial\rho - e^{2\rho}\big(1 + (x^-)^2\big)\partial x^+\Big)\,, \\ K^2 &= -\frac{1}{2}\Big(\partial x^- + 2x^-\partial\rho + e^{2\rho}\big(1 - (x^-)^2\big)\partial x^+\Big)\,, \\ K^3 &= i(e^{2\rho}x^-\partial x^+ - \partial\rho)\,, \end{aligned} \qquad\text{(B.8)}$$

and

$$\bar{K}^1 = \frac{i}{2}\Big(\bar{\partial}x^+ + 2x^+\bar{\partial}\rho - e^{2\rho}\big(1 + (x^+)^2\big)\bar{\partial}x^-\Big)\,,$$
$$\bar{K}^2 = -\frac{1}{2}\big(\bar{\partial}x^+ + 2x^+\bar{\partial}\rho + e^{2\rho}\big(1 - (x^+)^2\big)\bar{\partial}x^-\big)\,,$$
$$\bar{K}^3 = i(e^{2\rho}x^+\bar{\partial}x^- - \bar{\partial}\rho)\,,$$

(B.9)

respectively.

Now we can identify the charges of the combinations $K^1 \pm iK^2$ by using the Killing vector fields $j_{i\mathrm{R}}$, and $j_{i\mathrm{L}}$ dual to the forms j_{R}^i, and j_{L}^i. It turns out that

$$[j_{3\mathrm{R}}, j_{1\mathrm{R}} \pm ij_{2\mathrm{R}}] = \mp 2i(j_{1\mathrm{R}} \pm ij_{2\mathrm{R}})\,,$$
$$[j_{3\mathrm{L}}, j_{1\mathrm{L}} \pm ij_{2\mathrm{L}}] = \pm 2i(j_{1\mathrm{L}} \pm ij_{2\mathrm{L}})\,.$$

(B.10)

Therefore, to be consistent with the way we picked up the charges of the $\mathrm{SL}(2,\mathbb{R})$ currents in our construction of the $\mathcal{N} = 2$ SCFT algebra in appendix C, we should define $K^\pm = K^1 \mp iK^2$ and $\bar{K}^\pm = \bar{K}^1 \pm i\bar{K}^2$. We have

$$K^+ = ie^{2\rho}\partial x^+, \quad K^- = -i\Big(\partial x^- + 2x^-\partial\rho - e^{2\rho}(x^-)^2\partial x^+\Big)\,,$$

(B.11)

and

$$\bar{K}^+ = -ie^{2\rho}\bar{\partial}x^-, \quad \bar{K}^- = i\Big(\bar{\partial}x^+ + 2x^+\bar{\partial}\rho - e^{2\rho}(x^+)^2\bar{\partial}x^-\Big)\,.$$

(B.12)

Hence one obtains the following useful relation

$$\partial x^+\bar{\partial}x^- = \Phi_{1;-2,-2}^{sl}K^+\bar{K}^+\,,$$

(B.13)

where we used the semiclassical expressions for the $\mathrm{SL}(2,\mathbb{R})$ primaries

$$\Phi_{j;j+1,j+1}^{sl} = \frac{1}{g_{--}^{2(j+1)}}\,, \qquad \Phi_{j;-j-1,-j-1}^{sl} = \frac{1}{g_{++}^{2(j+1)}}\,,$$
$$\Phi_{j;j+1,-j-1}^{sl} = \frac{1}{g_{-+}^{2(j+1)}}\,, \qquad \Phi_{j;-j-1,j+1}^{sl} = \frac{1}{g_{+-}^{2(j+1)}}\,.$$

(B.14)

Notice that we use conventions where the $\mathrm{SL}(2,\mathbb{R})$ primary $\Phi_{j;m,\bar{m}}^{sl}$ at level $k + 2$ has conformal weight

$$\Delta = -\frac{j(j+1)}{k}\,.$$

(B.15)

We consider only the principal discrete series for which j is a real number, since we actually consider the universal cover of $\mathrm{SL}(2,\mathbb{R})$ in order to avoid any closed timelike curves, in the range

$$-\frac{1}{2} \leqslant j \leqslant \frac{k-1}{2}\,,$$

(B.16)

and m takes either the values $m = -j - j, -j - 2, \ldots$ or $m = j + 1, j + 2, \ldots$.

The action of the $\mathrm{SL}(2,\mathbb{R})$ affine currents $K^3, K^\pm$ on a primary field $\Phi_{j;m,\bar{m}}^{sl}$ is given by the OPEs

$$K^3(z)\Phi_{j;m,\bar{m}}^{sl}(w,\bar{w}) = \frac{m}{z-w}\Phi_{j;m,\bar{m}}^{sl}(w,\bar{w})\,,$$
$$K^\pm(z)\Phi_{j;m,\bar{m}}^{sl}(w,\bar{w}) = \frac{m \pm (j+1)}{z-w}\Phi_{j;m\pm 1,\bar{m}}^{sl}(w,\bar{w})\,.$$

(B.17)

The bosonic current algebra at level $k+2$ reads

$$
\begin{aligned}
K^3(z)K^3(w) &\sim -\frac{k+2}{2}\frac{1}{(z-w)^2}\,,\\
K^3(z)K^\pm(w) &\sim \pm\frac{K^\pm(w)}{(z-w)}\,,\\
K^+(z)K^-(w) &\sim \frac{k+2}{(z-w)^2} - 2\frac{K^3(w)}{z-w}\,,
\end{aligned}
\tag{B.18}
$$

while the corresponding fermions satisfy

$$
\begin{aligned}
\chi^3(z)\chi^3(w) &\sim -\frac{1}{z-w}\,,\\
\chi^+(z)\chi^-(w) &\sim \frac{1}{z-w}\,.
\end{aligned}
\tag{B.19}
$$

C The $\mathcal{N}=2$ superconformal algebra

We present here a realization of the $\mathcal{N}=2$ superconformal algebra in the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times \mathrm{U}(1)^4$ worldsheet theory. The energy-momentum tensor reads

$$
\begin{aligned}
T =\ & \frac{1}{k}\left[J^3 J^3 + \frac{1}{2}\left(J^+ J^- + J^- J^+\right) - K^3 K^3 + \frac{1}{2}\left(K^+ K^- + K^- K^+\right)\right] + \frac{1}{2}\sum_{a=1}^{4}\partial Y^a \partial Y^a\\
& -\frac{1}{2}\left[\psi^+\partial\psi^- + \psi^-\partial\psi^+ + \psi^3\partial\psi^3 + \chi^+\partial\chi^- + \chi^-\partial\chi^+ - \chi^3\partial\chi^3 + \sum_{a=1}^{4}\lambda^a\partial\lambda^a\right]\,,
\end{aligned}
\tag{C.1}
$$

while the $\mathcal{N}=2$ supercurrents take the form

$$
\begin{aligned}
G^+ &= \frac{1}{\sqrt{k}}\left[\left(J_{\mathrm{T}}^3 + K_{\mathrm{T}}^3\right)\left(\psi^3 - \chi^3\right) + \sqrt{2}\left(J^+\psi^- + K^-\chi^+\right)\right] + \hat{\lambda}^+\partial\hat{Y}^- + \tilde{\lambda}^+\partial\tilde{Y}^-\,,\\
G^- &= \frac{1}{\sqrt{k}}\left[\left(J_{\mathrm{T}}^3 - K_{\mathrm{T}}^3\right)\left(\psi^3 + \chi^3\right) + \sqrt{2}\left(J^-\psi^+ + K^+\chi^-\right)\right] + \hat{\lambda}^-\partial\hat{Y}^+ + \tilde{\lambda}^-\partial\tilde{Y}^+\,.
\end{aligned}
\tag{C.2}
$$

The $\mathrm{U}(1)$ R-charge current reads

$$
J_R = \frac{2}{k}\left(J_{\mathrm{T}}^3 + K_{\mathrm{T}}^3\right) - \psi^+\psi^- + \chi^+\chi^- + \psi^3\chi^3 + \hat{\lambda}^+\hat{\lambda}^- + \tilde{\lambda}^+\tilde{\lambda}^-\,,
\tag{C.3}
$$

where we have introduced the total $\mathrm{SU}(2)$ and $\mathrm{SL}(2,\mathbb{R})$ currents

$$
J_{\mathrm{T}}^3 = J^3 + \psi^+\psi^-\,, \qquad K_{\mathrm{T}}^3 = K^3 + \chi^+\chi^-\,.
\tag{C.4}
$$

Due to the presence of a time-like direction in the interacting non-linear sigma-model, the above generators turn out to be non-hermitian: the usual complex conjugation between G^+ and G^- does not hold.

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